Spherically symmetric scalar vacuum:
no-go theorems, black holes and solitons

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We prove some theorems characterizing the global properties of static, spherically symmetric configurations of a self-gravitating real scalar field φ in general relativity (GR) in various dimensions, with an arbitrary potential $V(\varphi)$, not necessarily positive-definite. The results are extended to sigma models, scalar-tensor and curvature-nonlinear theories of gravity. We show that the list of all possible types of space-time causal structure in the models under consideration is the same as the one for $\varphi = \mathrm{const}$, namely, Minkowski (or AdS), Schwarzschild, de Sitter and Schwarzschild — de Sitter, and all horizons are simple. In particular, these theories do not admit regular black holes with any asymptotics. Some special features of (2+1)-dimensional gravity are revealed. We give examples of two types of asymptotically flat configurations with positive mass in GR, still admitted by the above theorems: (i) a black hole with a nontrivial scalar field ("scalar hair") and (ii) a particlelike (solitonic) solution with a regular centre; in both cases, the potential $V(\varphi)$ must be at least partly negative. We also discuss the global effects of conformal mappings that connect different theories and illustrate such effects for solutions with a conformal scalar field in general relativity

1. Introduction

Vacuum spherically symmetric solutions to the Einstein equations are either Schwarzschild, or, if the cosmological constant is invoked, Schwarzschild — (anti-)de Sitter. All of them, except solutions with zero mass, contain curvature singularities at the centre.

A wider set of space-times is connected with the so-called false vacuum, i.e., the system with the action

$$S = \int d^4x \sqrt{-g} [R + (\partial\varphi)^2 - 2V(\varphi)] \qquad (1)$$

where R is the scalar curvature, $g = \det(g_{\mu\nu})$, φ is a scalar field, $(\partial\varphi)^2 = g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$, and the function $V(\varphi)$ is a potential. This action, with many particular forms of $V(\varphi)$, is conventionally used to describe the vacuum (sometimes interpreted as a variable cosmological term) in inflationary cosmology, for the description of growing vacuum bubbles, etc. In space-time regions where $\varphi = \mathrm{const}$ (if any), the corresponding potential $V(\varphi)$ behaves as a cosmological constant.

One might expect that the inclusion of a scalar field considerably widens the choice of possible qualitative behaviours of static, spherically symmetric configurations. Thus, one might seek among them regular black hole solutions, trying to substantiate the attractive idea of replacing the black hole (BH) singularity by a nonsingular vacuum core, which traces back to the 60s [1, 2] but remains in the scope of modern studies. Possible manifestations of regular BHs vary from fundamental particles to largest astrophysical objects and created universes [3, 4].

There are, however, very strong general restrictions that follow directly from the Einstein-scalar equations

due to (1). Thus, if $V \geq 0$, the only asymptotically flat BH solution is Schwarzschild, as follows from the well-known no-hair theorems (see Ref. [5] for a recent review). Another result concerns solitonic (particle-like) configurations with a regular centre and a flat asymptotic: if $V \geq 0$, then such a configuration cannot have a positive mass [6].

An attempt to construct a regular false vacuum BH was made in Ref. [7], with a potential having two slightly different minima, $V(\varphi_1) > V(\varphi_2) = 0$, the Schwarzschild metric and $\varphi \equiv \varphi_2$ outside the horizon, the de Sitter metric and $\varphi \equiv \varphi_1$ inside the horizon. It was claimed that a reasonable matching of the solutions was possible on the horizon despite a finite jump of φ. Gal'tsov and Lemos [8] showed that the piecewise solution of Ref. [7] cannot be described in terms of distributions and requires a singular matter source on the horizon. They proved [8] that asymptotically flat regular BH solutions are absent in the theory (1) with any nonnegative potential $V(\varphi)$ (the no-go theorem). Since, by the no-hair theorems, $\varphi = 0$ outside the horizon, in Ref. [8] it was sufficient to show that the Schwarzschild exterior cannot be smoothly matched to a regular BH interior with $\varphi \neq \mathrm{const}$.

Less is known if the asymptotic flatness and/or $V \geq 0$ assumptions are abandoned. Meanwhile, both assumptions are frequently violated in modern studies. Negative potential energy densities, in particular, the cosmological constant $V = \Lambda < 0$ giving rise to the anti–de Sitter (AdS) solution or AdS asymptotic, do not lead to catastrophes (if restricted below), are often treated in various aspects and quite readily appear from quantum effects like vacuum polarization.

We now continue the study of global properties of scalar-vacuum configurations begun in Ref. [9], which

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provided some essential restrictions on the possible behaviour of solutions of the theory (1) with arbitrary $V(\varphi)$ in 4 dimensions. This paper essentially generalizes and extends the results obtained there. Sec. 2 presents the field equations in D dimensions, $D \geq 3$. Sec. 3 contains the proofs of two restriction (no-go) theorems, which leave a very narrow spectrum of possible global space-time structures. According to Theorem 1, configurations like wormholes and horns are absent, while Theorem 3 shows that the variable scalar field adds nothing to the list of causal structures known for $\varphi = \mathrm{const}$: Minkowski (or AdS), Schwarzschild, de Sitter and Schwarzschild — de Sitter (not to be confused with the de Sitter — Schwarzschild structure having a de Sitter core, discussed in [10, 3]).

A conclusion much stronger than in Ref. [8], namely, the absence of regular BHs for *any* $V(\varphi)$ and with *any* asymptotic, then simply follows as a corollary.

It is shown that in (2+1)-dimensional theory no more than one simple horizon is possible, whereas in higher dimensions, including standard GR, there can be no more than two simple horizons.

We also consider the possible existence of geodesically complete space-times having curvature singularities at some of their asymptotics (the so-called remote singularities). Such singularities at finite r are ruled out for any configurations having a regular spatial asymptotic (Theorem 2).

Sec. 4 is devoted to 4-dimensional GR: we reproduces the no-hair theorem [11, 12] and the generalized Rosen theorem [6] for positive-semidefinite potentials.

Sec. 5 discusses extensions of Theorems 1–3 to some more general field models. We will consider (i) more general scalar field Lagrangians in GR, with an arbitrary dependence on the φ field and its gradient squared; (ii) multiscalar theories of sigma-model type; (iii) scalar-tensor theories of gravity; (iv) curvature-nonlinear (high-order) gravity with the Lagrangian of the form $f(R)$ where f is an arbitrary function. In items (iii) and (iv), conformal mappings are used to reduce the field equations to those due to (1) or (2).

In the Appendix we show that some opportunities allowed by the above theorems can be indeed realized by certain choices of $V(\varphi)$. Namely, we present particular examples of (i) an asymptotically flat BH with positive mass and $\varphi \neq \mathrm{const}$ and (ii) a particle-like solution with a regular centre, both being 4-dimensional and having positive masses. The potentials for such models are certainly (at least partly) negative. In addition, we discuss the well-known solution for a conformal scalar field in GR in order to illustrate how conformal mappings can affect the space-time global structure.

To conclude, with all theorems and examples, we now have, even without solving the field equations, rather a clear picture of what can and what cannot be expected from static, spherically symmetric scalar-vacuum configurations in various theories of gravity with various scalar field potentials.

Throughout the paper all statements apply to static, spherically symmetric configurations, and all relevant functions are assumed to be sufficiently smooth, unless otherwise indicated. The symbol □ will mark the end of a proof.

2. Field equations

We start with the D-dimensional generalization of the action (1)

$$S = \int d^D x \sqrt{|g|}[R + (\partial\varphi)^2 - 2V(\varphi)] \qquad (2)$$

The field equations due to (2) are

$$\nabla^\alpha\nabla_\alpha\varphi + V_\varphi = 0, \qquad (3)$$
$$R^\nu_\mu - \tfrac{1}{2}\delta^\nu_\mu R + T^\nu_\mu = 0, \qquad (4)$$

where $V_\varphi \equiv dV/d\varphi$, R^ν_μ is the Ricci tensor and T^ν_μ is the energy-momentum tensor of the φ field:

$$T^\nu_\mu = \varphi_{,\mu}\varphi^{,\nu} - \tfrac{1}{2}\delta^\nu_\mu(\partial\varphi)^2 + \delta^\nu_\mu V(\varphi).i \qquad (5)$$

Consider a static, spherically symmetric configuration, with the space-time structure

$$\mathbb{M}^{\overline{d}+2} = \mathbb{R}_t \times \mathbb{R}_\rho \times \mathbb{S}^{\overline{d}}, \qquad (6)$$

where $\mathbb{R}_t$ is the time axis, $\mathbb{R}_\rho \subset \mathbb{R}$ is the range of the radial coordinate ρ and $\mathbb{S}^{\overline{d}}$ ($\overline{d} = D-2$) is a $\overline{d}$-dimensional sphere. The metric can be written in the form

$$ds^2 = A(\rho)dt^2 - \frac{d\rho^2}{A(\rho)} - r^2(\rho)d\Omega^2_{\overline{d}}, \qquad (7)$$

where $d\Omega^2_{\overline{d}}$ is the linear element on $\mathbb{S}^{\overline{d}}$ of unit radius, and $\varphi = \varphi(\rho)$. (Without loss of generality, we suppose that large r corresponds to large ρ.) Accordingly, Eq. (3) and certain combinations of Eqs. (4) lead to

$$(Ar^{\overline{d}}\varphi')' = r^{\overline{d}}V_\varphi; \qquad (8)$$
$$(A'r^{\overline{d}})' = -(4/\overline{d})r^{\overline{d}}V; \qquad (9)$$
$$\overline{d}r''/r = -\varphi'^2; \qquad (10)$$
$$A(r^2)'' - r^2A'' + (\overline{d} - 2)r'(2Ar' - A'r)$$
$$= 2(\overline{d} - 1); \qquad (11)$$
$$\overline{d}(\overline{d} - 1)(1 - Ar'^2) - \overline{d}A'rr' = -Ar^2\varphi'^2 + r^2V, \qquad (12)$$

where the prime denotes $d/d\rho$. Only three of these five equations are independent: the scalar equation (8) follows from the Einstein equations, while Eq. (12) is a first integral of the others. Given a potential $V(\varphi)$, this is a determined set of equations for the unknowns r, A, φ.

The choice of the radial coordinate ρ such that $g_{tt}g_{\rho\rho} = -1$ is convenient for a number of reasons. First, we are going to deal with horizons, which correspond to zeros of the function $A(\rho)$. One can notice that such zeros are regular points of Eqs. (8)–(12), therefore one can jointly consider regions at both sides of a horizon.

Second, in a close neighbourhood of a horizon ρ varies (up to a positive constant factor) like manifestly well-behaved Kruskal-like coordinates used for an analytic continuation of the metric [13]. Third, with the same coordinate, horizons also correspond to regular points in geodesic equations [13]. Last but not least, this choice well simplifies the equations, in particular, (11) can be integrated, giving, for $\overline{d} \geq 2$,

$$B' \equiv \left(\frac{A}{r^2}\right)' = -\frac{2(\overline{d}-1)}{r^{\overline{d}+2}} \int r^{\overline{d}-2} d\rho. \tag{13}$$

3. No-go theorems and global structures

Now, our interest will be in the generic global behaviour of the solutions and the existence of BHs and globally regular configurations, in particular, regular BHs.

In these issues, a crucial role belongs to *Killing horizons,* regular surfaces where the Killing vector ∂_t is null. For the metric (7), a horizon $\rho = h$ is a sphere of nonzero radius $r = r_h$ where $A = 0$. The space-time regularity implies the finiteness of T_μ^ν, so that V and $A\varphi'^2$ are finite at $\rho = h$. The latter does not forbid $\varphi' \to \infty$ since $A(h) = 0$. If we, however, additionally (and reasonably) assume that the metric functions, including $r(\rho)$, are at least C^2-smooth at $\rho = h$, then r'' is finite, and $|\varphi'| < \infty$ follows from (10).

The horizon is *simple* or *multiple* (or *higher-order*) according to whether the zero of the function $A(\rho)$ is simple or multiple. Thus, the Schwarzschild horizon is simple while the extreme Reissner-Nordström one is double.

As usual, we shall call the space-time regions where $A > 0$ and $A < 0$ *static* (R) and *nonstatic* (T) regions, respectively. A simple or odd-order horizon separates a static region from a nonstatic one, whereas an even-order horizon separates two regions of the same nature. On the construction of Carter-Penrose diagrams, characterizing the causal structure of arbitrary static 2-dimensional space-times [such as the (t, ρ) section of (7)] see Refs. [15, 16] and more recent and more comprehensive papers [17, 18].

3.1. Regular models without a centre?

The first restriction is that such configurations as wormholes, horns or flux tubes do not exist under our assumptions.

For the metric (7), a (traversable, Lorentzian) *wormhole* is, by definition, a configuration with two asymptotics at which $r \to \infty$, hence with $r(\rho)$ having at least one regular minimum. A *horn* is a region where, as ρ tends to some value ρ^*, $r(\rho) \neq$ const and $g_{tt} = A$ have finite limits while the length integral $l = \int d\rho/A$ diverges. In other words, a horn is an infinitely long $(\overline{d} + 1)$-dimensional "tube" of finite radius, with the clock rate remaining finite everywhere. Such "horned

particles" were, in particular, discussed as possible remnants of black hole evaporation [14]. Lastly, a *flux tube* is a configuration with $r = $ const.

Theorem 1. *The field equations due to (2) do not admit (i) solutions where the function $r(\rho)$ has a regular minimum, (ii) solutions describing a horn, and (iii) flux-tube solutions with $\varphi \neq $ const.*

Proof. By its geometric meaning (the "area function", radius of a coordinate sphere), $r(\rho) \geq 0$, therefore Eq. (10) gives $r'' \leq 0$, which rules out regular minima. The same equation leads to $\varphi = $ const as soon as $r = $ const. Thus items (i) and (iii) have been proved.

Suppose now that there is a horn. Then, by the above definition, A has a finite limit whereas $l \to \infty$ as $\rho \to \rho^*$. This is only possible if $\rho^* = \pm\infty$. Under these circumstances, the left-hand side of Eq. (13) vanishes at the "horn end", $\rho \to \rho^* = \pm\infty$, whereas its right-hand side tends to infinity. This contradiction proves item (ii). $\square$

Due to the local nature of the proof, this statement means the absence of wormhole throats or horns in solution having any large r behaviour — flat, de Sitter or any other, or having no large r asymptotic at all.

It also follows that the full range of the ρ coordinate covers all values of r, from the centre ($\rho = \rho_c$, $r(\rho_c) = 0$), regular or singular, to infinity, unless (which is not excluded) there is a singularity at finite r due to a "pathological" choice of the potential.

The latter opportunity deserves attention since, being singular at zero or finite r, the space-time may in principle be still geodesically complete. In other words, any geodesic can only reach the singularity at an infinite value of its canonical parameter. No freely moving particle can then attain such a singularity (to be called a *remote singularity* in finite proper time. Examples of remote singularities are known in solutions of 2-dimensional gravity [19]).

Let us find out whether such remote singularities can appear in systems under consideration. The integral of geodesic equations for our metric (7) can be written as

$$\left(\frac{d\rho}{d\sigma}\right)^2 + kA + L^2\frac{A}{r^2} = E^2, \tag{14}$$

where σ is the canonical parameter, L and E are the conserved angular momentum and energy of a moving particle and $k = 1, 0, -1$ for timelike, null and spacelike geodesics, respectively. Non-circular motion ($r \neq$ const) may be parametrized with the coordinate ρ, and the canonical parameter can be found from (14) in the form

$$\sigma = \pm \int \frac{d\rho}{\sqrt{E^2 - AL^2/r^2 - kA}}. \tag{15}$$

This integral can diverge at some ρ for all values of the constants of motion E and L simultaneously only if $\rho \to \pm\infty$. Due to $r'' \leq 0$, this can happen either at
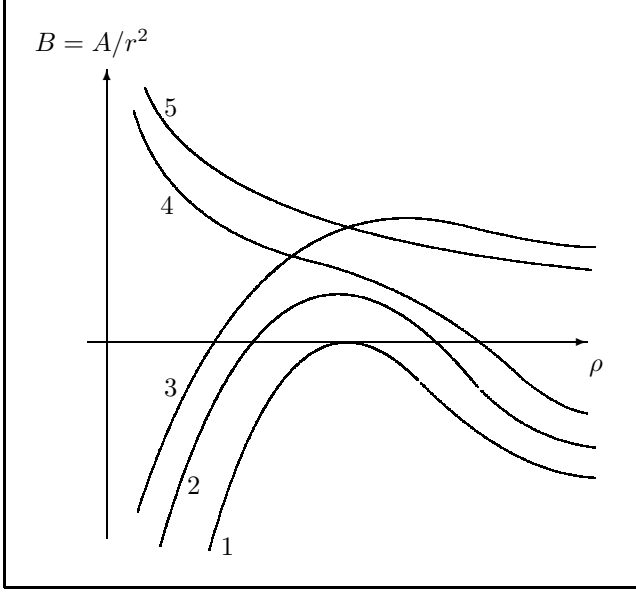
Figure 1: Generic qualitative behaviours of the function $B(\rho)$ determining the causal structure of space-time: 1 — no static region; 2 — Schwarzschild – de Sitter; 3 — Schwarzschild; 4 — de Sitter; 5 — Minkowski/AdS

infinity ($r \to \infty$), or at finite $\rho = \rho_s$, but then $r < r(\rho_s)$ for all values of ρ. We can state the following:

Theorem 2 (on remote singularities).. *If a solution to Eqs. (8)–(12) has a spatial asymptotic ($r \to \infty$), it cannot contain a remote singularity at $r < \infty$.*

3.2. Global structures

Now, taking into account Theorem 1, the global space-time structure corresponding to any particular solution is unambiguously determined (up to identification of isometric surfaces, if any) by the disposition of static ($A > 0$) and nonstatic ($A < 0$) regions. The following theorem severely restricts the choice of horizon dispositions in the theory under study.

Theorem 3 (on the horizons). *Consider solutions of the theory (2), $D \geq 4$, with the metric (7) and $\varphi = \varphi(\rho)$. Let there be a static region $a < \rho < b \leq \infty$. Then:*

(i) *all horizons are simple;*
(ii) *no horizons exist at $\rho < a$ and at $\rho > b$.*

Proof. It follows from Eq. (11) that if $\rho = h$ is a horizon of order higher than 1, i.e., $A(h) = A'(h) = 0$, then $A''(h) = -2(\overline{d} - 1)/r^2 < 0$, i.e., it is a double horizon separating two T regions. The function $I(\rho) = \int r^{\overline{d}-2} d\rho$ is monotonically increasing, $I'(\rho) > 0$, while Eq. (13) implies $I(h) = 0$, hence $I < 0$ and $B' > 0$ at $\rho < h$; likewise, $I > 0$ and $B' < 0$ at $\rho > h$. Therefore both A and B are negative for all $\rho \neq h$, i.e., there is no static region — item (i) is proved.

Consider now the boundary $\rho = a$ of the static region. If $r(a) = 0$, it is the centre; be it regular or singular, it is then the left boundary of the range of ρ. If $r(a) \neq 0$, then it is a horizon; since $a < \rho < b$ is a static region, $B'(a) > 0$, then by (13) $I(a) < 0$. By monotonicity, $I(\rho<a) < 0$, so that $B'(\rho<a) > 0$. This means that B cannot return to zero to the left of a, i.e., there is no horizon. In a similar way one can verify that horizons are absent to the right of b. □

According to Theorem 3, there can be no more than two horizons, and the list of possible global structures is the same as the one well-known for constant φ:

[TR]: Schwarzschild/Tangherlini (curve 3 in Fig. 1),
[RT]: de Sitter (curve 4),
[R]: Minkowski or AdS (curve 5),
[TRT]: Schwarzschild – de Sitter (curve 2), and
[TT], [T]: spacetimes without static regions (curve 1 and still below).

The R and T letters in brackets show the sequence of static and nonstatic regions, ordered from center to infinity. The center is generically singular. The only possible nonsingular solutions have either Minkowski/AdS or de Sitter structures, and, in particular, solitonlike asymptotically flat solutions are not excluded.

Corollary. *The theory (1) does not admit static, spherically symmetric, regular BHs.*

Indeed, such a BH, with any large r behavior, must have static regions at small and large r, separated by at least two simple or one double horizon (in the above notation, the structure must be [RTR] or [RR] or more complex). This is impossible according to Theorem 3: such configurations are simply absent in the above list.

More generally, if spatial infinity is static, there is at most one simple horizon; the same is true if the centre is static.

Special case: (2+1)-dimensional gravity

In 3 dimension we have $\overline{d} = 1$, and integration of (11) leads to an expression simpler than (13):

$$B' \equiv (A/r^2)' = C/r^3, \qquad C = \text{const.} \qquad (16)$$

In Theorem 1, items (i) and (iii) hold due to Eq. (10), as before. Still, the proof of item (ii) does not work: a horn is possible if, in (16), $C = 0$. Though, due to $r'' < 0$, the horn radius r^* is the maximum of $r(\rho)$, so that a horned configuration has no large r asymptotic.

By virtue of (16), B' has a constant sign coinciding with sign C, and, instead of Theorem 3, we have a still more severe restriction:

Theorem 3a.. *A static, circularly symmetric configuration in the theory (2), $D = 3$, has either no horizon or one simple horizon.*

Accordingly, the list of possible global structures is even shorter than the previous one: the structures [TT]

and [TRT] are now absent, and the only structures possessing horizons are Schwarzschild-like (possible when $C > 0$) and de Sitter-like (possible when $C < 0$). Purely static and purely nonstatic solutions are certainly possible as well; in particular, when $C = 0$, (16) gives $A = C_1 r^2$ — a global R or T region depending on the sign of the constant C_1.

4.   4-dimensional GR: more restrictions

The above theorems did not use any assumptions on the asymptotic behaviour of the solutions or the shape and even sign of the potential. Let us now mention some more specific but also very significant results for positive-semidefinite potentials.

Consider, for simplicity, $D = 4$. The behaviour of the field functions at a regular centre and at a flat asymptotic (if, certainly, these are present in the solution).

A **regular centre**, where $r = 0$, implies a finite time rate and local spatial flatness. This means that at some finite $\rho = \rho_c$

$$Ar'^2 \to 1, \qquad A = A_c + O(r^2), \qquad (17)$$

where $A_c = A(\rho_c)$ and $r'(\rho_c)$ are finite and positive. Moreover, the values of V, φ and φ' should be finite there. Then from (10) and (13) one obtains:

$$r''(\rho_c) = 0; \qquad \rho_0 = \rho_c. \qquad (18)$$

At a **flat asymptotic**, the metric should behave as the Schwarzschild one with a certain mass M, while φ should tend to a finite value. The corresponding conditions are

$$\rho \to \infty; \qquad r' \to 1; \qquad A(\rho) = 1 - \frac{2M}{\rho} + O(\rho^{-2});$$

$$\varphi' = o(\rho^{-3/2}); \qquad V = o(\rho^{-3}); \qquad (19)$$

The last two requirements follow from the field equations.

One of the restrictions is the well-known no-hair theorem:

Theorem 4 (no-hair). *Suppose $V \geq 0$. Then the only asymptotically flat BH solution to Eqs. (8)–(12) in the range (h, ∞) (where $\rho = h$ is the event horizon) comprises the Schwarzschild metric, $\varphi = $ const and $V \equiv 0$.*

This theorem was first proved by Bekenstein [11] for the case of $V(\varphi)$ without local maxima and was later refined for any $V \geq 0$ and for certain more general Lagrangians — see e.g. Ref. [5] for proofs and references.

Let us give, for completeness, a proof similar to that of Ref. [12] using our ρ coordinate.

Proof. Eqs. (8), (10) and (12) make in possible to prove the identity

$$\frac{d}{d\rho}\left[\frac{1}{r'}(2r^2V - Ar^2\varphi'^2)\right] = 4rV + r\varphi'^2\left(\frac{1}{r'^2} + A\right). (20)$$

Suppose that $\rho = h$ is an event horizon and $\rho = \infty$ is flat infinity. It follows from (10) that $r(\rho) \leq \rho + \mathrm{const}$. Therefore, recalling the horizon properties mentioned in Sec. 2, one can assert that h is finite while $r(h) = r_h$ and $r'(h) \geq 1$ are finite and positive. Integrating (20) from h to ∞ and taking into account (19), one obtains

$$-\frac{2}{r'(h)}\,r_h^2 V(\varphi(h)) = \int\limits_h^\infty r\left[4V + \varphi'^2\left(\frac{1}{r'^2} + A\right)\right]d\rho. (21)$$

Since here the l.h.s. is nonpositive while both terms in the integrand are nonnegative, Eq. (21) only holds if V and φ' are identically zero for $\rho \in (h, \infty)$. The asymptotically flat metric is then necessarily Schwarzschild. □

Another restriction can be called the generalized Rosen theorem (G. Rosen [20] studied similar restrictions for flat-space nonlinear field configurations):

Theorem 5 [6]. *An asymptotically flat solution with positive mass M and a regular centre is impossible if $V(\varphi) \geq 0$.*

Proof. Let us integrate Eq. (9) from the centre ($\rho = \rho_c$) to infinity:

$$A'r^2\Big|_{\rho_c}^\infty = -2\int_{\rho_c}^\infty r^2 V\, d\rho. \qquad (22)$$

In an asymptotically flat metric, $A(\rho)$ behaves at large ρ as $1 - 2M/\rho$, where M is the Schwarzschild mass in geometric units, and $r = \rho + O(1)$, therefore the upper limit of $A'r^2$ equals $2M$. At a regular centre $r = 0$ and, as is easily verified, $A' = 0$, so the lower limit is zero. Consequently,

$$M = -\int_{\rho_c}^\infty r^2 V\, d\rho. \qquad (23)$$

Thus a positive mass M requires an at least partly negative potential $V(\varphi)$. □

D-dimensional generalizations of Theorems 3 and 4 are possible but will not be considered here.

The above theorems leave some opportunities of interest, in particular:

1. BHs with $\varphi \neq$ const, potentials $V(\varphi) \geq 0$ but non-flat large r asymptotics;

2. asymptotically flat BHs with $\varphi \neq$ const but at least partly negative potentials $V(\varphi)$;

3. asymptotically flat particlelike solutions (solitons) with positive mass but at least partly negative potentials $V(\varphi)$.

That such solutions do exist, one can prove by presenting examples. Such examples are known for item 1 due to the paper by Chan, Horne and Mann [21], where, among other results, BHs with non-flat asymptotics were found for the Liouville ($V = 2\Lambda \, \mathrm{e}^{2b\varphi}$) and double Liouville ($V = 2\Lambda_1 \, \mathrm{e}^{2b_1\varphi} + 2\Lambda_2 \, \mathrm{e}^{2b_2\varphi}$) potentials, where the Λ's and b's are positive constants.

We will give special analytical solutions to Eqs. (8)–(12) for $\overline{d} = 2$, exemplifying items 2 (Appendix A)[1] and 3 (Appendix B). Unlike Ref. [21], where special solutions were sought for by making the ansatz $r(\rho) \propto \rho^N$, $N = \mathrm{const}$ (in our notation), we will use the following approach. Suppose $V(\varphi)$ is one of the unknowns. Then our set of equations is underdeterminate, and we can choose one of the unknowns arbitrarily trying to provide the proper behaviour of the solution. Thus, one can choose a particular function $r(\rho)$: assigning it arbitrarily and substituting into (13), by single integration we obtain $A(\rho)$, after which $\varphi(\rho)$ and $V(\rho)$ are determined from (10) and (9), respectively. Thus $V(\varphi)$ is obtained in a parametric form; it can be made explicit if $\varphi(\rho)$ resolves with respect to ρ.

The purpose of giving these examples is to merely demonstrate the existence of such kinds of solutions, therefore the physical meaning of the potentials obtained will not be discussed.

5. Generalizations

5.1. More general Lagrangians in GR. Sigma models

One can notice that Theorems 1–3 actually rest on two Einstein equations, (10) and (11), which in turn follow from the properties of the energy-momentum tensor: $T_t^t - T_\rho^\rho \geq 0$, which expresses the validity of the null energy condition for systems with the metric (7). This, through the corresponding Einstein equation, leads to $r'' \leq 0$. Eq. (11), which leads to Theorem 3, follows from the property

$$T_t^t = T_\theta^\theta \tag{24}$$

where θ is any of the coordinate angles that parametrize the sphere $\mathbb{S}^{\overline{d}}$.

Therefore these three theorems hold for all kinds of matter whose energy-momentum tensors satisfy these two conditions.

Consider, for instance, the following action, more general than (1):

$$S = \int d^D x \sqrt{-g}[R + F(I, \varphi)] \tag{25}$$

where $I = (\partial\varphi)^2$ and $F(I, \varphi)$ is an arbitrary function. The scalar field energy-momentum tensor is

$$T_\mu^\nu = \frac{\partial F}{\partial I}\varphi_{,\mu}\varphi^{,\nu} + \tfrac{1}{2}\delta_\mu^\nu F(\varphi). \tag{26}$$

In the static, spherically symmetric case, Eq. (24) holds automatically due to $\varphi = \varphi(\rho)$, while the null energy condition holds as long as $\partial F / \partial I \geq 0$, which actually

means that the kinetic energy is nonnegative. Under this condition, all Theorems 1–3 are valid for the theory (25). Otherwise Theorem 3 alone holds; it correctly describes the ρ dependence of A and consequently the possible horizons disposition, but the situation is more complex due to possible non-monotonicity of $r(\rho)$.

Another important and frequently discussed class of theories is the class of the so-called sigma models, where a set of N scalar fields $\varphi = \{\varphi^a\}$, $a = \overline{1, N}$ are considered as coordinates of a target space with a certain metric $G_{ab} = G_{ab}(\varphi)$. The scalar vacuum action is then written in the form

$$S_\sigma = \int d^D x \sqrt{|g|}[R + G_{ab}g^{\mu\nu}\partial_\mu\varphi^a\partial_\nu\varphi^b - 2V(\varphi)] \tag{27}$$

where, in general, G_{ab} and V are arbitrary functions of N variables, but in practice they possess symmetries that follow from the nature of specific systems.

It is easily seen that, provided the metric $G_{ab}(\varphi)$ is positive-definite, Theorems 1–3 for static, spherically symmetric configurations are valid as before.

If G_{ab} is not positive-definite, or if some of φ^a are allowed to be imaginary, only Theorem 3 holds.

5.2. Scalar-tensor and higher-order gravity

Other extensions of the present results concern theories connected with (2) and (25) via φ-dependent conformal transformations, such as theories with nonminimally coupled scalar fields (e,g., scalar-tensor theories, STT) and nonlinear gravity (e.g., with the Lagrangian function $f(R)$).

Above all, it should be noted that if a space-time $\mathbb{M}[g]$ with the metric (7) is conformally mapped into another space-time $\overline{\mathbb{M}}[\overline{g}]$, equipped with the same coordinates, according to the law

$$g_{\mu\nu} = F(\rho)\overline{g}_{\mu\nu}, \tag{28}$$

then it is easily verified that a horizon $\rho = h$ in $\mathbb{M}$ passes into a horizon of the same order in $\overline{\mathbb{M}}$, (ii) a centre ($r = 0$), an asymptotic ($r \to \infty$) and a remote singularity in $\mathbb{M}$ passes into a center, an asymptotic and a remote singularity, respectively, in $\overline{\mathbb{M}}$ if the conformal factor $F(\rho)$ is regular (i.e., finite, at least C^2-smooth and positive) at the corresponding values of ρ. A regular centre passes into a regular centre and a flat asymptotic to a flat asymptotic under evident additional requirements, but we will not concentrate on them here.

The general (Bergmann-Wagoner-Nordtvedt) STT action in D dimensions can be written as follows:

$$S_{\mathrm{STT}} = \int d^D x \sqrt{|g|}[f(\phi)R + h(\phi)(\partial\phi)^2 - 2U(\phi) + L_m], \tag{29}$$

where f, h and U are functions of the scalar field ϕ and L_m is the matter Lagrangian. The metric $g_{\mu\nu}$ here corresponds to the so-called Jordan conformal frame.

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[1]Note that two recent examples of exact solutions describing BHs with scalar hair found by Lechtenfeld and co-authors [22] concern scalar fields with an anomalous sign of kinetic energy, which are of certain interest but are beyond the scope of this paper.

The standard transition to the Einstein frame [23],

$$g_{\mu\nu} = F(\varphi)\overline{g}_{\mu\nu}, \qquad F = f^{-2/(D-2)}, \qquad (30)$$

$$\frac{d\varphi}{d\phi} = \frac{\sqrt{|l(\phi)|}}{f(\phi)}, \qquad l(\phi) \overset{\text{def}}{=} fh + \frac{D-1}{D-2}\left(\frac{df}{d\phi}\right)^2, \quad (31)$$

removes the nonminimal scalar-tensor coupling expressed in a ϕ-dependent coefficient before R. Putting $L_m = 0$ (vacuum), one can write the action (29) in terms of the new metric $\overline{g}_{\mu\nu}$ and the new scalar field φ as follows (up to a boundary term):

$$S_{\text{E}} = \int d^D x \sqrt{|\overline{g}|}[R_{\text{E}} + \eta_l(\partial\varphi)^2 - 2V(\varphi)], \qquad (32)$$

where R_{E} and $(\partial\varphi)^2$ are calculated using $\overline{g}_{\mu\nu}$,

$$V(\varphi) = \eta_f F^2(\varphi)\, U(\phi), \qquad (33)$$

and $\eta_{l,f}$ are sign factors:

$$\eta_l = \text{sign}\, l(\phi), \qquad \eta_f = \text{sign}\, f(\phi). \qquad (34)$$

Note that $\eta_l = -1$ corresponds to the so-called anomalous STT, with a wrong sign of scalar field kinetic energy, while $\eta_f = -1$ means that the effective gravitational constant in the Jordan frame is negative. So the normal choice of signs is $\eta_{l,f} = 1$.

The action (32) obviously coincides with (2) up to the coefficient η_l. Therefore Eq. (24) holds, and we can assert that, for static, spherically symmetric configurations, Theorem 3 is valid for the Einstein-frame metric $\overline{g}_{\mu\nu}$.

Theorems 1 and 2 hold for $\overline{g}_{\mu\nu}$ only in the "normal" case $\eta_l = 1$; let us adopt this restriction.

The validity of the theorems for the Jordan-frame metric $g_{\mu\nu}$ depends on the nature of the conformal mapping (30) between the space-times $\mathbb{M}[g]$ (Jordan) and $\overline{\mathbb{M}}[\overline{g}]$ (Einstein). There are four variants:

I. $\mathbb{M} \longleftrightarrow \overline{\mathbb{M}}$,
II. $\mathbb{M} \longleftrightarrow (\overline{\mathbb{M}}_1 \subset \overline{\mathbb{M}})$,
III. $(\mathbb{M}_1 \subset \mathbb{M}) \longleftrightarrow \overline{\mathbb{M}}$,
IV. $(\mathbb{M}_1 \subset \mathbb{M}) \longleftrightarrow (\overline{\mathbb{M}}_1 \subset \overline{\mathbb{M}})$,

where $\longleftrightarrow$ denotes a diffeomorphism preserving the metric signature. The last three variants are possible if the conformal factor F vanishes or blows up at some values of ρ, which then mark the boundary of $\mathbb{M}_1$ or $\overline{\mathbb{M}}_1$.

Theorem 3 on horizon dispositions is obviously valid in $\overline{\mathbb{M}}$ in cases I and II. In case III or IV, the whole space-time $\overline{\mathbb{M}}$ or its part is put into correspondence to only a part $\mathbb{M}_1$ of $\mathbb{M}$, and, generally speaking, anything, including additional horizons, can appear in the remaining part $\mathbb{M}_2 = \mathbb{M} \setminus \mathbb{M}_1$ of the Jordan-frame space-time.

Theorem 1 cannot be directly transferred to $\mathbb{M}$ in any case except the trivial one, $F = \text{const}$. It is only possible to assert, without specifying $F(\varphi)$, that wormholes as global entities are impossible in $\mathbb{M}$ in cases I and II if the conformal factor F is finite in the whole range of ρ, including the boundary values. Indeed, if we suppose that there is such a wormhole, it will immediately follow that there are two large r asymptotics and a minimum of $r(\rho)$ between them even in $\overline{\mathbb{M}}$, in contrast to Theorem 1 which is valid there.

Theorem 2 also evidently holds in $\mathbb{M}$ in cases I and II if the conformal factor F is regular in the whole range of ρ, including the boundary values.

Another class of theories conformally equivalent to (2) is the so-called higher-order gravity with the vacuum action

$$S_{\text{HOG}} = \int d^D x \sqrt{|g|} f(R) \qquad (35)$$

where f is a function of the scalar curvature R calculated for the metric $g_{\mu\nu}$ of a space-time $\mathbb{M}$. The conformal mapping $\mathbb{M}[g] \mapsto \overline{\mathbb{M}}[\overline{g}]$ with

$$g_{\mu\nu} = F(R)\overline{g}_{\mu\nu}, \qquad F = f_R^{-2/(D-2)}, \qquad (36)$$

(where, in accord with the weak field limit $f \sim R$ ar small R, we assume $f(R) > 0$ and $f_R \overset{\text{def}}{=} df/dR > 0$), transforms the action (35) into (2) with

$$\varphi = \frac{1}{2}\sqrt{\frac{D-1}{D-2}}\log f_R, \qquad (37)$$

$$V(\varphi) = f_R^{-D/(D-2)}(Rf_R - f). \qquad (38)$$

The field equations due to (35) after this substitution turn into the field equations due to (2).

All the above observations on the validity of Theorems 1–3 in STT equally apply to curvature-nonlinear gravity.

It should be noted that most of papers which use such conformal mappings assume, explicitly or implicitly, a one-to-one correspondence between $\mathbb{M}$ and $\overline{\mathbb{M}}$ (variant I). Meanwhile, the most general case is given by variant IV, where the mapping only connects subsets of the respective manifolds. This situation is similar to a transformation between coordinate frames in a single manifold.

An exactly soluble example with a conformally coupled scalar field in GR, when the mapping follows variant III and there appear horizons or wormhole throats outside M_1, is presented in Appendix C.

To conclude, with all these theorems and examples, we now have, without solving the field equations, a more or less clear picture of what can and what cannot be expected from static, spherically symmetric scalar-vacuum configurations in various theories of gravity with various scalar field potentials.

Appendix

A. An asymptotically flat black hole

Let us introduce an arbitrary constant length scale a and the dimensionless quantities

$$x = \rho/a, \qquad R(x) = r/a; \qquad x_0 = \rho_0/a. \qquad (\text{A.1})$$

The quantities φ and A are already dimensionless and are expressed in terms of the known (prescribed) function $R(x)$ as follows:

$$\psi(x) \stackrel{\text{def}}{=} \varphi/\sqrt{2} = \pm \int \sqrt{-\frac{R_{xx}}{R}}\,dx, \qquad (A.2)$$

$$A(x) = 2R^2 \int_x^\infty \frac{\bar{x} - \bar{x}_0}{R^4(\bar{x})}\,d\bar{x}. \qquad (A.3)$$

where the subscript x denotes d/dx. Eq. (A.3) is written in such a form that $A(\infty) = 1$ if $R(x)$ behaves at large x as $x + \text{const} + o(1)$, thus providing asymptotic flatness.

Choosing $R(x)$ in the form

$$R(x) = \sqrt{x^2 - 1}, \qquad (A.4)$$

we obtain (restricting to the $+$ sign in (A.2)):

$$\psi(x) = \text{Arcoth}\,x \equiv \frac{1}{2}\log\frac{x+1}{x-1}, \qquad (A.5)$$

$$A(x) = 1 - xx_0 + x_0\frac{x^2-1}{2}\log\frac{x+1}{x-1}. \qquad (A.6)$$

Substituting these expressions into (9) and finally inserting $x = \coth\psi$, we arrive at an explicit form of the dimensionless potential $U = a^2 V$:

$$\begin{aligned} U &= \frac{x_0}{2(x^2-1)}\left[6x + (1-3x^2)\log\frac{x+1}{x-1}\right] \\ &= x_0\left[\frac{3}{2}\sinh(2\psi) - \psi\cosh(2\psi) - 2\psi\right]. \end{aligned} \qquad (A.7)$$

This is a BH solution in case $x_0 > 0$, and there is a naked singularity in case $x_0 < 0$. It is easy to verify that the Schwarzschild mass M (according to the asymptotic form $A = 1 - 2M/\rho + ...$ at large ρ) is given by $3M = \rho_0 = ax_0$. Therefore the BH branch of this solution corresponds to a positive mass, just as in the Schwarzschild solution.

The field φ ranges from zero at infinity to infinity at the singular centre, $x = 1$. There is one simple, Schwarzschild-like horizon ($A = 0$), in full agreement with the general treatment of Sec. 2.

The functions $\psi(x)$, $A(x)$, $U(x)$ and $U(\psi)$ are plotted in Fig. 2. One can notice that, in the present example, the potential is everywhere negative and is unbounded below; though, its finiteness in the outer region of the BH indicates that the latter property is not necessary for obtaining BH solutions but is a consequence of our simple choice of $R(x)$.

B. A particlelike solution

Creation of such an example is a more difficult task than the previous one since one now has to take into account two boundary conditions, at the centre ($\rho = \rho_c$, $r(\rho_c) = 0$) and at the infinity. Let us, using the same notations (A.1), make the following choice:

$$R^2(x) = \frac{1}{u^2}\frac{\tanh(u+c)}{\tanh c}, \qquad u \equiv \frac{1}{x}, \qquad (B.1)$$
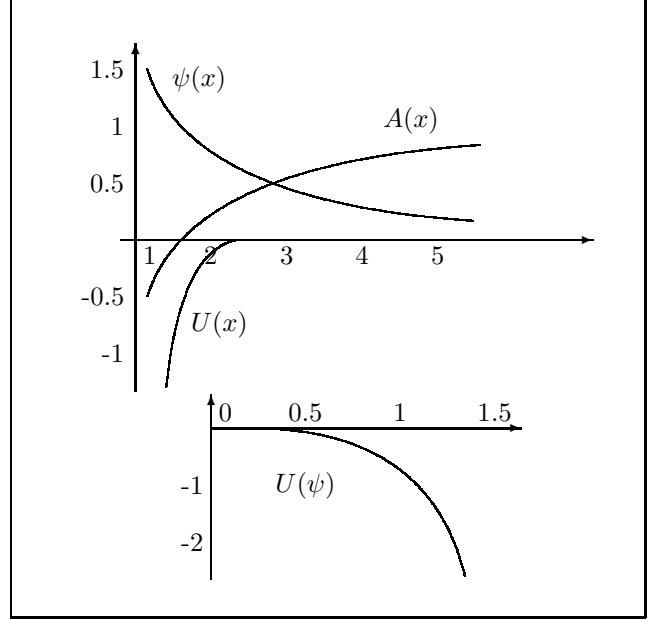

Figure 2: The functions $\psi(x)$, $A(x)$, $U(x)$ and $U(\psi)$ for the example of a black hole solution with $x_0 = 2$.

where c is a positive constant, and put $x_0 = 0$ in Eq. (A.3). As before, one can explicitly find the quantities A and φ in terms of x (or u) from Eqs. (A.2) and (A.3):

$$\sqrt{2}\varphi = \sqrt{3}\arctan\frac{Z}{\sqrt{3}} - \frac{1}{2}\log\frac{Z+1}{Z-1},$$

$$Z \equiv \sqrt{2\cosh[2(u+c)] - 1}, \qquad (B.2)$$

$$\begin{aligned} A = A_0\tanh(u+c)\Big[&1 - \frac{2}{uA_0} + \frac{2}{u}\frac{\sinh u}{\sinh c\cosh(u+c)} \\ &+ \frac{2}{u^2}\log\frac{\sinh(u+c)}{\sinh c}\Big], \end{aligned} \qquad (B.3)$$

where $A_0 = \tanh c$ is the value of A at the centre, $u = \infty$. One can verify that the regular centre conditions (17) hold at $u = \infty$.

The expression for $U(x) = a^2 V$, obtained from Eq. (9), can also be found explicitly, but is too long to be presented here.

The field φ varies between two finite values at large and small x, and the potential V in this range is smooth and finite too. The corresponding plots are presented in Fig. 3.

The Schwarzschild mass, calculated as before, is connected with c or A_0 as follows:

$$M = \frac{a}{6}\frac{1 - A_0^2}{A_0} = \frac{a}{3\sinh(2c)}. \qquad (B.4)$$

where a is the arbitrary length scale introduced in (A.1).

Thus we have constructed a family of solitonic solutions with an arbitrary positive mass.
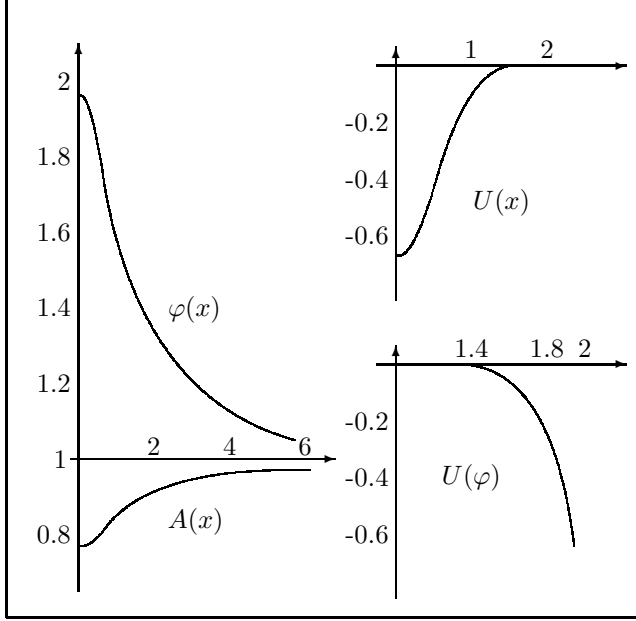
Figure 3: The functions $\varphi(x)$, $A(x)$, $U(x)$ and $U(\varphi)$ for the example of a particlelike solution with $c = 1$.

C. Conformal scalar field in GR: black holes and wormholes

Conformal scalar field in GR can be viewed as a special case of STT, such that, in Eq. (29), $D = 4$ and

$$f(\phi) = 1 - \phi^2/6, \qquad h(\phi) = 1, \qquad U(\phi) = 0. \quad (\text{C.1})$$

After the conformal mapping

$$g_{\mu\nu} = F(\varphi)\overline{g}_{\mu\nu}, \qquad F(\varphi) = \cosh^2(\varphi/\sqrt{6}), \qquad (\text{C.2})$$
$$\phi = \sqrt{6}\tanh(\varphi/\sqrt{6}), \qquad (\text{C.3})$$

we obtain the action (2) with $D = 4$ and $V \equiv 0$. The latter describes a minimally coupled massless scalar field in GR, and the corresponding static, spherically symmetric solution is well-known: it is the Fisher solution [24]. It is convenient to write it using the harmonic radial coordinate u specified by the condition [25] $|g_{uu}| = g_{tt}g_{\theta\theta}^2$ (in the previous notation, u behaves as $1/r$ at large r):

$$ds_{\text{E}}^2 = \text{e}^{-2mu}dt^2 - \frac{k^2\,\text{e}^{2mu}}{\sinh^2(ku)}\left[\frac{k^2du^2}{\sinh^2(ku)} + d\Omega^2\right],$$
$$\varphi = \sqrt{6}C(u + u_0), \qquad (\text{C.4})$$

where the subscript "E" stands for the Einstein frame, m (the mass), C (the scalar charge), $k > 0$ and u_0 are integration constants, and k is expressed in terms of m and C:

$$k^2 = m^2 + 3C^2. \qquad (\text{C.5})$$

The previously used coordinate ρ, corresponding to the metric (7), $D = 4$, is connected with u as follows:

$$\rho = k\cot(ku), \qquad (\text{C.6})$$

and the metric in terms of ρ has the form

$$ds_{\text{E}}^2 = \left(\frac{\rho - k}{\rho + k}\right)^{m/k}dt^2$$
$$- \left(\frac{\rho + k}{\rho - k}\right)^{m/k}\left[d\rho^2 + (\rho^2 - k^2)d\Omega^2\right]. \quad (\text{C.7})$$

This solution is asymptotically flat at $u \to 0$ ($\rho \to \infty$), has no horizon when $C \neq 0$ (as should be the case according to the no-hair theorem) and is singular at the centre ($u \to \infty$, $\rho \to k$, $\varphi \to \infty$). It turns into the Schwarzschild solution when $C = 0$.

The "Jordan-frame" solution is described by the metric $ds^2 = F(\phi)ds_{\text{E}}^2$ and the ϕ field according to (C.2). It is the conformal scalar field solution [26, 28], whose properties are more diverse and can be enumerated as follows (putting, for definiteness, $m > 0$ and $C > 0$):

1. $C < m$. The metric behaves qualitatively as in the Fisher solution: it is flat at $u \to 0$, and both g_{tt} and $r^2 = |g_{\theta\theta}|$ vanish at $u \to \infty$ — a singular attracting centre. A difference is that here the scalar field is finite: $\phi \to \sqrt{6}$.

2. $C > m$. Instead of a singular centre, at $u \to \infty$ one has a repulsive singularity of infinite radius: $g_{tt} \to \infty$ and $r^2 \to \infty$. Again $\varphi \to \sqrt{6}$ as $u \to \infty$.

3. $C = m$. In this case the metric and the scalar field are regular at $u = \infty$, and a continuation across this regular sphere may be achieved using a new coordinate, e.g.,

$$y = \tanh(mu). \qquad (\text{C.8})$$

The solution acquires the form

$$ds^2 = (1 + yy_0)^2\left[\frac{dt^2}{(1 + y)^2}\right.$$
$$\left. - \frac{m^2(1 + y)^2}{y^4(1 - y_0)^2}(dy^2 + y^2d\Omega^2)\right],$$
$$\phi = \sqrt{6}\frac{y + y_0}{1 + yy_0}, \qquad (\text{C.9})$$

where $y_0 = \tanh(mu_0)$. The range $u \in \mathbb{R}_+$, which describes the whole manifold $\overline{\mathbb{M}}$ in the Fisher solution, corresponds in $\mathbb{M}$ to the range $0 < y < 1$, describing only a region of $\mathbb{M}$, the manifold of the solution (C.9). The properties of the latter depend on the sign of y_0 [25]. In all cases, $y = 0$ corresponds to a flat asymptotic, where $\phi \to \sqrt{6}y_0$, $|y_0| < 1$.

3a: $y_0 < 0$. The solution is defined in the range $0 < y < 1/|y_0|$. At $y = 1/|y_0|$, there is a naked attracting central singularity: $g_{tt} \to 0$, $r^2 \to 0$, $\phi \to \infty$.

3b: $y_0 > 0$. The solution is defined in the range $y \in \mathbb{R}_+$. At $y \to \infty$, we find another flat spatial infinity, where $\phi \to \sqrt{6}/y_0$, $r^2 \to \infty$ and g_{tt} tends to a finite limit. This is a wormhole solution, found for the first

time by one of the authors [25] and recently discussed by Barcelo and Visser [27].

3c: $y_0 = 0$, $\phi = \sqrt{6}y$, $y \in \mathbb{R}_+$. In this case it is helpful to pass to the conventional coordinate r, substituting $y = m/(r - m)$. The solution

$$ds^2 = (1 - m/r)^2 dt^2 - \frac{dr^2}{(1 - m/r)^2} - r^2 d\Omega^2,$$

$$\phi = \sqrt{6}m/(r - m) \tag{C.10}$$

is the well-known BH with a conformal scalar field [26, 28], which seems to violate the no-hair theorem. The infinite value of ϕ at the horizon $r = m$ does not make the metric singular since, as is easily verified, the energy-momentum tensor remains finite there.

The whole case 3 belongs to variant III in the classification of Sec. 5.2, and the horizon in case 3c is situated in the region $\mathbb{M}_2 = \mathbb{M} \setminus \mathbb{M}_1$, where the action of the no-hair theorem cannot be extended.

In case 3b, the second spatial infinity and even the wormhole throat ($y = 1/\sqrt{y_0}$) are situated in $\mathbb{M}_2$, illustrating the inferences of Sec. 5.

However, in case 2, where the mapping is type I by the same classification ($\mathbb{M} \longleftrightarrow \overline{\mathbb{M}}$), there appears a minimum of $r(u)$ in the metric $g_{\mu\nu}$ (C.2), and r even blows up at large u. This is connected with blowing up of the conformal factor F. Recall that in Sec. 5.2 the absence of another spatial infinity was only guaranteed under the finiteness condition for the conformal factor in the whole range of the radial coordinate, including its boundary values: we see that this condition is indeed essential.

The simple example of the conformal field thus illustrates the possible nontrivial consequences of conformal mappings.

Acknowledgements

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